DAY 2 (MAY 29, 2025)

There are 5 problems for Day 1 of the competition. Each problem is worth 7 points.

Problem 1 (Byron Walden)**.** Define a sequence $\{b_n\}$ by putting $b_1 = 1$, $b_2 = 3$, $b_3 = 2$ and for all $n \geq 4$, letting b_n be the average of the smaller two numbers from the set $\{b_{n-1},\ b_{n-2},\ b_{n-3}\}$. Find

$$\lim_{n\to\infty} b_n.$$

Solution: Noting that $b_1 < b_3 = \frac{b_1+b_2}{2} < b_2$ for the basis, we show inductively that

$$b_{3n-2} < b_{3n} = \frac{b_{3n-2}+b_{3n-1}}{2} < b_{3n-1}$$

for all $n \geq 1$. Suppose that $b_{3n-2} < b_{3n} = \frac{b_{3n-2}+b_{3n-1}}{2} < b_{3n-1}$ holds for some $n \geq 1$. Then

$$b_{3n+1} = \frac{b_{3n-2}+b_{3n}}{2},$$

and the correct ordering for the latest three elements of the sequence is $b_{3n+1} < b_{3n} < b_{3n-2}$. This gives

$$b_{3n+2} = \frac{b_{3n+1}+b_{3n}}{2},$$

and the correct ordering for the latest three elements: $b_{3n+1} < b_{3n+2} < b_{3n}$. So finally,

$$b_{3n+3} = \frac{b_{3n+1}+b_{3n+2}}{2},$$

and the correct ordering for the latest three: $b_{3n+1} < b_{3n+3} < b_{3n+2}$, proving the hypothesis of our induction.

Having established $b_{3n-2} < b_{3n} = \frac{b_{3n-2}+b_{3n-1}}{2} < b_{3n-1}$, we can write $d_n = b_{3n} - b_{3n-2} = b_{3n-1} - b_{3n}$. So

$$2d_{n+1} = b_{3n+2} - b_{3n+1} = \frac{b_{3n+1}+b_{3n}}{2} - b_{3n+1} = \frac{b_{3n}+b_{3n+1}}{2} = \frac{b_{3n} - \frac{b_{3n-2}+b_{3n}}{2}}{2} = \frac{b_{3n}-b_{3n-2}}{4} = \frac{d_n}{4},$$

whence $d_{n+1} = \frac{d_n}{8}$ and as $d_1 = 1$, we see $d_n = \frac{1}{8^{n-1}}$. As $b_{3n+1} = b_{3n+3} - \frac{1}{8^n}$ is the average of $b_{3n-2} = b_{3n} - \frac{1}{8^{n-1}}$ and b_{3n}, we get the equation $b_{3n+3} - \frac{1}{8^n} = b_{3n} - \frac{1}{2\cdot 8^{n-1}}$ or

$$b_{3n} - b_{3n+3} = \frac{1}{2\cdot 8^{n-1}} - \frac{1}{8^n} = \frac{3}{8^n}.$$

From there,

$$2 - b_{3m} = b_3 - b_{3m} = \sum_{n=1}^{m-1}(b_{3n} - b_{3n+3}) = \sum_{n=1}^{m-1}\frac{3}{8^n} = \frac{\frac{3}{8} - \frac{3}{8^m}}{1-\frac{1}{8}} = \frac{3 - \frac{3}{8^{m-1}}}{7}.$$

Letting m approach ∞ yields

$$\lim_{m\to\infty} b_{3m} = 2 - \frac{3}{7} = \frac{11}{7}.$$

As d_m approaches 0 as m approaches ∞ and b_{3m-1} and b_{3m-2} are $b_{3m} \pm d_m$, the other terms of the sequence of b_n's must approach $\frac{11}{7}$ as well.

Problem 2 (Greta Panova). A fly sits on the side AB of the triangle ABC with side lengths a, b, c, then flies to BC, then to side AC and returns to the point it started from. Suppose $\angle A, \angle B, \angle C < 90°$, what is the minimal possible distance the fly traveled expressed as a rational function of a, b, c?

Solution: Let the points on the sides be X, Y, Z along the route of the fly. Using the reflection principle, for fixed X and Z, the minimal distance $XY + YZ$ occurs when Y is such that $\angle XYB = \angle CYZ =: y$. Similarly we obtain $\angle AXZ = \angle YXB =: x$ and $\angle AZX = \angle CZY =: z$. Let $\alpha = \angle A, \beta = \angle B, \gamma = \angle C$.

Let I be the incenter of XYZ, i.e. where the angle bisectors meet. We thus get that $\angle IXA = \angle IXZ + \angle ZXA = \angle IXY + \angle YXB$ and thus have to be $90°$, and similarly for the others. The lines $AB \perp IX$, $BC \perp IY$, $AC \perp IZ$. Thus, for a given triangle XYZ, there is a unique triangle ABC, and I is its orthocenter. Since $XBYI$ lie on a circle ($\angle X + \angle Y = 180°$), we have $90° - y = \angle IBX = \angle ZBA = 90° - \alpha$. So $y = \alpha$ and similarly $x = \gamma$ and $z = \beta$. Then $AXZ \sim ACB$ with proportionality constant r_1, $XBY \sim CBA$ with proportionality constant r_2 and $YCZ \sim ACB$ proportionality constant r_3. Then $AX = r_1 b$, $XB = r_2 a$, and since they sum up to $AB = c$, we get the system of equations

$$\begin{aligned} r_1 b + r_2 a &= c \\ r_1 c + r_3 a &= b \\ r_2 c + r_3 b &= a \end{aligned}$$

and we solve for the rs:

$$r_1 = \frac{c^2 + b^2 - a^2}{2cb}, r_2 = \frac{a^2 + c^2 - b^2}{2ac}, r_3 = \frac{a^2 + b^2 - c^2}{2ab}$$

and thus

$$XY + YZ + ZX = r_1 a + r_2 b + r_3 c = \frac{2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4}{2abc}$$

Problem 3 (Greta Panova)**.** Find all pairs of integers p, q, such that

$$2(p-1)^3 + q = 8pq - 1.$$

Solution: Write the equation as $2(p-1)^3 - 8q(p-1) - 7q + 1 = 0$, so $p-1|7q-1$ and let $7q-1 = a(p-1)$, $x = p-1$, so we get

$$2x^2 - 8(ax+1)/7 - a = 0. \Longrightarrow 14x^2 - 8ax - 7a - 8 = 0.$$

Solving this as a quadratic equation we get

$$D = (4a)^2 + 98a + 8 * 14 = (4a + \frac{49}{4})^2 - 49^2/16 + 8 * 14$$

D has to be a square of a rational number. After clearing denominators we are left with

$(16a+49)^2 - (3*16+1)^2 + 7*16^2 = u^2, \Longrightarrow (16a-49-u)(16a-49+u) = 609 = 3*203 = 3*7*29$

Note that if $16a + 49 - u = b$ and $16a + 49 + u = c$, then $32|b + c - 98$, so $32|b + c - 2$. We have the following possibilities for $b + c$ modulo 32:

1. $b + c \equiv \pm(1 + 609) = \pm(2) \pmod{32}$.
2. $b + c \equiv \pm(3 + 7 * 29) = \pm(-18) \pmod{32}$
3. $b + c \equiv \pm(7 + 3 * 29) = \pm(-2) \pmod{32}$
4. $b + c \equiv \pm(29 + 3 * 7) = \pm(18) \pmod{32}$.

The only possibilities are thus in cases 1 and 3.

From case 1 we have (assuming $u > 0$) that $b = 1$ and $c = 609$, so $u = 304$ and $a = 16$. This gives $x = 10$ and $p = 11$ and $q = 23$ as a solution.

From case 3 we have that $b = -3 * 29$ and $c = -7$, so $u = 40$ but $a < 0$. This is not possible since $p, q > 1$, and so we obtained the only possible solution.

Problem 4 (Harm Derksen)**.** The function $\operatorname{sinc}(x)$ is defined on $\mathbb{R}$ by $\operatorname{sinc}(x) = \sin(x)/x$ for $x \neq 0$ and $\operatorname{sinc}(0) = 1$. Prove that

$$\prod_{n=1}^{\infty} \operatorname{sinc}\left(\frac{x}{2n-1}\right) = \prod_{n=1}^{\infty} \cos\left(\frac{x}{2n}\right)$$

for all $x \in \mathbb{R}$.

Solution: We first show that all the relevant infinite products converge. The functions $\log(x)$ and $\operatorname{sinc}(x)$ are analytic and by computing the first Taylor coefficients we see that $\log(\operatorname{sinc}(x)) = -x^2/6 + O(x^4)$. So the sum $\sum_{n=m}^{\infty} \log(\operatorname{sinc}(x/n))$ converges absolutely for $|x| < m\pi/2$. By exponetiating, we see that the infinite product $\prod_{n=1}^{\infty} \operatorname{sinc}(x/n)$ converges for all x. Similarly, the products $\prod_{n=1}^{\infty} \operatorname{sinc}(x/(2n-1))$, $\prod_{n=1}^{\infty} \operatorname{sinc}(x/(2n))$ and $\prod_{n=1}^{\infty} \cos(x/(2n))$ converge for all x. Suppose that $x = \pi n$ where n is an integer. If $n = 0$, then the products $\prod_{n=1}^{\infty} \operatorname{sinc}(x/(2n-1))$ and $\prod_{n=1}^{\infty} \cos(x/(2n))$ are both equal to 1. If n is nonzero, then $\operatorname{sinc}(x) = \cos(x/(2n)) = 0$ and both products are equal to 0. Now assume that x is not an integer multiple of π. Then $\sin(x/(2n))$ is nonzero for all n and the product $\prod_{n=1}^{\infty} \operatorname{sinc}(x/(2n))$ converges to a nonzero real number.

We have

$$\prod_{n=1}^{\infty} \operatorname{sinc}\left(\frac{x}{2n-1}\right) = \prod_{n=1}^{\infty} \frac{\sin\left(\frac{x}{2n-1}\right)}{\left(\frac{x}{2n-1}\right)} = \frac{\prod_{n=1}^{\infty} \frac{\sin\left(\frac{x}{n}\right)}{\left(\frac{x}{n}\right)}}{\prod_{n=1}^{\infty} \frac{\sin\left(\frac{x}{2n}\right)}{\left(\frac{x}{2n}\right)}} = \prod_{n=1}^{\infty} \frac{\sin\left(\frac{x}{n}\right)}{2\sin\left(\frac{x}{2n}\right)}.$$

Using the angle doubling formula $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ with $\alpha = x/(2n)$, we see that the right-hand side is equal to $\prod_{n=1}^{\infty} \cos(x/(2n))$.

Problem 5 (Harm Derksen)**.** We define the Fibonacci sequence by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Find all real polynomials $p(x)$ of degree 3 for which there exists infinitely many positive integers n such that both n and $p(n)$ are Fibonacci numbers.

Solution: Let $\phi = (1+\sqrt{5})/2$ (the golden ratio). The roots of the polynomial $x^2 - x - 1$ are ϕ and $-\phi^{-1}$. So $\phi^2 = \phi + 1$ and multiplying by ϕ^{n-2} gives $\phi^n = \phi^{n-1} + \phi^{n-2}$ for all $n \in \mathbb{Z}$. So if $A_n = \phi^n$ satisfies the recurrence relation

$$A_n = A_{n-1} + A_{n-2} \tag{1}$$

Similarly, $A_n = (-\phi)^n$ satisfies (1). Because $A_n = F_n$ satisfies the recurrence and the recurrence is linear, $A_n = F_n - 5^{-1/2}\phi^n + 5^{-1/2}(-\phi)^n$ satisfies (1). It is easily verified that $A_1 = A_2 = 0$ and using (1) and induction we get $A_n = 0$ for all n. So for all $n \geq 1$ we have

$$F_n = \frac{\phi^n - (-\phi^{-1})^n}{\sqrt{5}}.$$

(This formula is well-known.)

Suppose $p(x) = ax^3 + bx^2 + cx + d$ has the property that $p(F_n)$ is a Fibonacci number for infinitely many positive integers n. Suppose $p(x) = ax^3 + bx^2 + cx + d$ and there exists infinite sequences $1 \leq n_1 < n_2 < n_3 < \cdots$ and $m_1, m_2, m_3, \ldots$ such that $p(F_{n_k}) = F_{m_k}$ for all $k \geq 1$. We have $\lim_{k\to\infty} F_{m_k}/\phi^{m_k} = 5^{-1/2}$ and $\lim_{k\to\infty} p(F_{n_k})/\phi^{3n_k} = a5^{-3/2}$. As $k \to \infty$

$$\phi^{m_k - 3n_k} = \frac{\phi^{m_k}}{F_{m_k}} \cdot \frac{p(F_{n_k})}{\phi^{3n_k}}$$

converges to $5^{1/2} \cdot a5^{-3/2} = a/5$. This shows that $m_k - 3n_k$ is eventually constant. So we have $F_{3n+r} = p(F_n)$ for infinitely many positive integers n.

Case 1: Suppose that $F_{3n+r} = p(F_n)$ for infinitely many even positive integers n. For any such even integer n we have

$$\begin{aligned}5^{-1/2}(\phi^{3n+r} - \phi^{-3n-r}) &= p(F_n) = aF_n^3 + bF_n^2 + cF_n + d = \\ &= a5^{-3/2}(\phi^{3n} - 3\phi^n + 3\phi^{-n} - \phi^{-3n}) + b5^{-1}(\phi^{2n} - 2 + \phi^{-2n}) + c5^{-1/2}(\phi^n - \phi^{-n}) + d.\end{aligned}$$

This means that we have an equality of Laurent polynomials

$$\begin{aligned}&5^{-1/2}(\phi^r x^3 - \phi^{-r}x^{-3}) = \\ &\quad = a5^{-3/2}(x^3 - 3x + 3x^{-1} - x^{-3}) + b5^{-1}(x^2 - 2 + x^{-2}) + c5^{-1/2}(x - x^{-1}) + d.\end{aligned} \tag{2}$$

because we have equality for infinitely many values of x of the form ϕ^n. Comparing coefficients of x^3 and x^{-3} we get $r^{-1/2}\phi^r = a5^{-3/2} = 5^{-1/2}\phi^{-r}$, so we must have $r = 0$ and $a = 5$. comparing the coefficents of other powers of x gives $b = d = 0$ and $c = 3$. In this case we get $p(x) = 5x^3 + 3x$. With these choices of a, b, c, d, r, plugging $x = \phi^n$ into (2) gives $p(F_n) = F_{3n}$ whenever n is even.

Case 2: Suppose that $F_{3n+r} = p(F_n)$ for infinitely many odd integers n. For any such integer n we have

$$\begin{aligned}5^{-1/2}(\phi^{3n+r} + \phi^{-3n-r}) &= p(F_n) = aF_n^3 + bF_n^2 + cF_n + d = \\ &= a5^{-3/2}(\phi^{3n} + 3\phi^n + 3\phi^{-n} + \phi^{-n}) + b5^{-1}(\phi^{2n} + 2 + \phi^{-2n}) + c5^{-1/2}(\phi^n + \phi^{-n}) + d.\end{aligned}$$

Similar reasoning as in case 1 gives $r = 0$, $b = d = 0$, $a = 5$ and $c = -3$. So we get $p(x) = 5x^3 - 3x$ and $p(F_n) = F_{3n}$ for all odd positive integers n.