## DAY 1 (MAY 28, 2025)

*There are 5 problems for Day 1 of the competition. Each problem is worth 7 points.*

**Problem 1** (Byron Walden)**.** There are two circles in the $xy$-plane centered on the $y$-axis which are tangent to both the parabola $y = x^2$ and the line $y = 2025$. Determine the lengths of the circles' diameters.

**Solution:** Suppose $(0, c)$ is the center of a circle tangent to the parabola at the points $(\pm a, a^2)$. The line through $(0, c)$ and $(a, a^2)$ is perpendicular to the tangent to the parabola at $(a, a^2)$, and thus has slope $-\frac{1}{2a}$. So,

$$\frac{c - a^2}{0 - a} = -\frac{1}{2a},$$

which implies $c = a^2 + \frac{1}{2}$. Consequently, the radius of the circle is given by

$$r = \sqrt{(a - 0)^2 + \left(a^2 - (a^2 + \frac{1}{2}\right)^2} = \sqrt{a^2 + \frac{1}{4}}.$$

Then $c = a^2 + \frac{1}{2} = r^2 + \frac{1}{4}$ and the circle has horizontal tangents at

$$y = c \pm r = r^2 + \frac{1}{4} \pm r = (r \pm \frac{1}{2})^2.$$

So we need to solve $(r \pm \frac{1}{2})^2 = 2025 = 45^2$. The solutions are $r = 44.5$ and $45.5$ and the desired diameter lengths are $2(44.5) = 89$ and $2(45.5) = 91$.

**Problem 2** (Greta Panova)**.** Find all possible values of $gcd(a^{2m}+1, a^n+1)$, where $a, m, n$ are positive integers and $n$ is odd.

**Solution:** Suppose that there exists an odd prime $p$ such that $p|a^{2m}+1$ and $p|a^n+1$. Let $g$ be order of $a$ modulo $p$, i.e., the smallest integer such that $a^g \equiv 1 \pmod{p}$. We have that $a^{2n} \equiv 1 \pmod{p}$ and $a^{4m} \equiv 1 \pmod{p}$, so we must have $g|2n$ and $g|4m$. Since $a^n \not\equiv 1 \pmod{p}$, we must have that $g \not| n$ so $g = 2g_1$, where $g_1|n$. Similarly, $2g_1 \not| 2m$ but $g_1|2m$, so we must have $g_1 = 2g_2$, where $g_2|m$. Then $g = 4g_2|2n$, so $2g_2|n$, which is a contradiction with $n$ being odd.

Now suppose that $4|a^{2m}+1$ and $4|a^n+1$. Then $(a^m)^2 \equiv -1 \pmod{4}$, which is impossible since $x^2 \equiv 0, 1 \pmod{4}$ for all integers $x$.

Thus, the only possible values for the $gcd$ are 1 (when $a = 2$) and 2 (when $a = 1$).

**Problem 3** (Byron Walden)**.** Two points are chosen randomly – independently with uniform probability – from a semicircular arc with radius 1. A third point is chosen randomly – independently with uniform probability – from the diameter that connects the endpoints of the arc. What is the expected value of the area of the triangle with the three chosen points as its vertices?

**Solution:** We choose $\alpha$, $\beta$ from $[0, \pi]$ and $x$ from $[-1, 1]$. In order to find the expected area $E$, define $A(\alpha, \beta, x)$ to be the area of the triangle with vertices $e^{i\alpha}$, $e^{i\beta}$, $x$. Then

$$E = \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \int_{-1}^1 A(\alpha, \beta, x) dx d\alpha d\beta.$$

Replacing $x$ be $-x$ in the integral would leave $E$ unchanged, so we can equivalently compute

$$E = \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \int_{-1}^1 \frac{A(\alpha, \beta, x) + A(\alpha, \beta, -x)}{2} dx d\alpha d\beta.$$

But using similar triangles (among many possible arrangements) shows the distance from 0 to the line through $e^{i\alpha}$ and $e^{i\beta}$ is the average of the distances from $x$ to $-x$ to that line. So

$$\begin{aligned}
E &= \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \int_{-1}^1 A(\alpha, \beta, x) dx d\alpha d\beta \\
&= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi A(\alpha, \beta, 0) d\alpha d\beta \\
&= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin|\beta - \alpha| d\alpha d\beta \qquad \text{(sine rule)} \\
&= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \sin(\beta - \alpha) d\alpha d\beta \qquad \text{(symmetry)} \\
&= \frac{1}{\pi^2} \int_0^\pi \cos(\beta - \alpha)\Big|_{\alpha=0}^{\beta} d\alpha d\beta \\
&= \frac{1}{\pi^2} \int_0^\pi (1 - \cos\beta) d\beta \\
&= \frac{1}{\pi^2} (\beta - \sin\beta)\Big|_0^\pi \\
&= \frac{1}{\pi^2} (\pi - 0) = \frac{1}{\pi}.
\end{aligned}$$

**Problem 4** (Harm Derksen)**.** Define a sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 2$, $a_2 = 5$ and $a_{n+2} = f(a_n, a_{n+1})$ for all $n \geq 1$, where

$$f(x, y) = 5(x + y) + 2\sqrt{6x^2 + 15xy + 6y^2}.$$

Show that $a_n$ is an integer for all $n \geq 1$.

**Solution:** It is clear that the sequence $a_1, a_2, \ldots$ is increasing and positive. We verify that $a_2 = 5(2 + 5) + 2\sqrt{324} = 35 + 2 \cdot 18 = 71$ is an integer. If $z = f(x, y)$ then we have $z - 5x - 5y = 2\sqrt{6x^2 + 15xy + 6y^2}$. Squaring gives

$$25x^2 + 25y^2 + z^2 + 50xy - 10xz - 10yz = 24x^2 + 60xy + 24y^2.$$

After rearranging the terms we get

$$x^2 + y^2 + z^2 = 10(xy + xz + yz).$$

For all $n \geq 0$

$$a_n^2 + a_{n+1}^2 + a_{n+2}^2 = 10(a_n x_{n+1} + a_n a_{n+2} + a_{n+1} a_{n+2}) \tag{1}$$

and

$$a_{n+1}^2 + a_{n+2}^2 + a_{n+3}^2 = 10(a_{n+1} a_{n+2} + a_{n+1} a_{n+3} + a_{n+2} x_{n+3}) \tag{2}$$

Subtracting (1) from (2) gives

$$\begin{aligned}(a_{n+3} - a_n)(a_{n+3} + a_n) &= a_{n+3}^2 - a_n^2 = \\ &10(a_{n+1} a_{n+3} + a_{n+2} a_{n+3} - a_n a_{n+1} - a_n a_{n+2}) = 10(a_{n+3} - a_n)(a_{n+1} + a_{n+2})\end{aligned}$$

Since $a_{n+3} > a_n$ we can divide by $a_{n+3} - a_n$ and get $a_{n+3} + a_n = 10(a_{n+2} + a_{n+1})$ and $a_{n+3} = 10a_{n+2} + 10a_{n+1} - a_n$ for $n \geq 1$. Because $a_1, a_2, a_3$ are integers, it follows that $a_n$ is a positive integer for all $n \geq 1$ by induction. The sequence is increasing and positive because $0 < a_1 < a_2 < a_3$ and by induction we get $a_{n+3} - a_{n+2} = 9a_{n+2} + 9a_{n+1} + (a_{n+1} - a_n) > a_{n-1} - a_n > 0$ for all $n \geq 1$.

**Problem 5** (Harm Derksen)**.** Show that for every positive integer $n$ there exist nonnegative integers $p, q$ and integers $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q \geq 2$ such that

$$n = \frac{(a_1^3 - 1)(a_2^3 - 1)\cdots(a_p^3 - 1)}{(b_1^3 - 1)(b_2^3 - 1)\cdots(b_q^3 - 1)}.$$

**Solution:** Let $G = \{x \in \mathbb{Q} \mid x > 0\}$ be the multiplicative group of positive real numbers and $H$ be the subgroup of $G$ generated by the elements of the form $n^3 - 1$ where $n$ is an integer $\geq 2$. Using the factorizations $n^3 - 1 = (n-1)(n^2+n+1)$, $n^3+1 = (n+1)(n^2-n+1)$ and $n^6 - 1 = (n^3 - 1)(n^3 + 1)$ we get

$$\frac{(n+1)^6 - 1}{\big((n+1)^3 - 1\big)\big(n^3 - 1\big)} = \frac{(n+1)^3 + 1}{n^3 - 1} = \frac{(n+2)\big((n+1)^2 - (n+1) + 1\big)}{(n-1)(n^2+n+1)} = \frac{n+2}{n-1}.$$

If we substitute $n = 3k+1$ we get

$$\frac{\big(((3k+2)^2)^3 - 1\big)}{((3k+2)^3 - 1)((3k+1)^3 - 1)} = \frac{3k+3}{3k} = \frac{k+1}{k}.$$

This shows $(k+1)/k$ lies in $H$ for all $k \geq 1$. It follows that $m = \prod_{k=1}^{m-1} \left(\frac{k+1}{k}\right) \in H$ for every positive integer $m$.